

A convex approach to differential inclusions with prox-regular sets: stability analysis and observer design*

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1 Introduction

We consider in this paper the problem relying on the existence and stability of solutions of the following differential inclusion, given in a Hilbert space H ,

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)) & \text{ae } t \geq 0 \\ x(0; x_0) = x_0 \in C, \end{cases} \quad (1)$$

where N_C is the normal cone to an r -uniformly prox-regular closed subset C of H . The dynamical system driven by the set C is subject to a Lipschitz continuous perturbation mapping f defined on H . By a solution of (1) we mean a function $x(\cdot; x_0) : [0, +\infty) \rightarrow H$, with $x(0; x_0) = x_0$, which is absolutely continuous on every interval $[0, T]$, $T > 0$, and satisfies (1) for almost every (ae) $t \geq 0$; hence, in particular, $x(t) \in C$ for all $t \geq 0$. Indeed, such a solution is necessarily Lipschitz continuous on such intervals $[0, T]$ (see Theorem 4.1). Differential inclusion (1) appears in the modeling of many concrete problems in economy,

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unilateral mechanics, electrical engineering as well as optimal control (see eg. [1], [15], [22] and references therein.)

It was recently shown in [14] and [15] that (1) has one and only one (absolutely continuous) solution, which satisfies the imposed initial condition. These authors employed a regularization approach based on the Moreau-Yoshida approximation, and use the nice properties of uniform prox-regularity to show that the approximate scheme converges to the required solution. In this way, such an approach repeats those arguments of approximation ideas which, previously, were extensively used in the setting of differential inclusions with maximal monotone operators.

Problems dealing with the stability of solutions of (1), namely the characterization of weakly lower semi-continuous Lyapunov pairs and functions, have been developed in [16] following the same strategy, also based on Moreau-Yoshida approximations. Most of works on these problems use indeed this natural approximation approach; see, e.g. [14, 15, 16].

In this paper, at a first glance we provide a different, but quite direct, approach to tackle this problem. We prove that problem (1) can be equivalently written as a differential inclusion given in the current Hilbert setting under the form

$$\begin{cases} \dot{x}(t) \in g(x(t)) - A(x(t)) & \text{ae } t \in [0, T], \\ x(0; x_0) = x_0 \in \text{dom} A, \end{cases} \quad (2)$$

where $A : H \rightrightarrows H$ is an appropriate maximal monotone operator defined on H , and $g : H \rightarrow H$ is a Lipschitz continuous mapping. Then, it will be sufficient to apply the classical theory of maximal monotone operators ([7]; see, also, [4, 5]) to analyze the existence and the stability of solutions for differential inclusion (1). The concept of invariant sets will be the key tool to go back and forth between inclusions (1) and (2). Invariant sets with respect to differential inclusions governed by maximal monotone operators have been studied and characterized in [4, 5]. Other references for invariant sets, also referred to as viable sets, and the related theory of Lyapunov stability are [6, 8, 13, 18] among others. We shall also provide different criteria for the so-called α -Lyapunov pairs of lower semi-continuous functions to extend some of the results given in [4, 5, 16] to the current setting. It is worth to observe that the assumption of uniformly prox-regularity is required to obtain global solutions of (1), which are defined on the whole interval $[0, T]$. However, our analysis also works in the same way when the set C is prox-regular at x_0 rather than being a uniformly prox-regular set; but, in this case, we only obtain a local solution defined around x_0 .

This paper is organized as follows. After giving the necessary notations and preliminary results in Section 2, we review and study in Section 3 different aspects of the theory of differential inclusions governed by maximal monotone operators, including the existence of solutions, and we provide a stability results dealing with the invariance of closed sets with respect to such differential inclusions. In Sections 4, we provide the new proof of the existence of solutions for differential inclusions involving normal cones to r -uniformly prox-regular sets. Section 5 is devoted to the characterization of lower semi-continuous α -

Lyapunov pairs and functions. Inspired from the recent paper [23], we give in section 6 an application of our result to a Luenberger-like observer.

2 Preliminaries and examples

2.1 Preliminary results

In this paper, H is a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and an associated norm $\| \cdot \|$. The strong and weak convergences in H are denoted by \rightarrow and \rightharpoonup , resp. We denote by $B(x, \rho)$ the closed ball centered at $x \in H$ of radius $\rho > 0$, and particularly we use \mathbb{B} for the closed unit ball. The null vector in H is written θ .

Given a set $S \subset H$, by $\text{co}S$, $\text{cone}S$ and \overline{S} we respectively denote the *convex hull*, the *conic hull* and the *closure* of S . The dual cone of S is the set

$$S^* := \{x^* \in H \mid \langle x^*, x \rangle \leq 0 \text{ for all } x \in S\}.$$

The *indicator* and the *distance functions* are respectively given by

$$I_S(x) := 0 \text{ if } x \in S; +\infty \text{ if not, } d_S(x) := \inf\{\|x - y\| : y \in S\}$$

(in the sequel we shall adopt the convention $\inf \emptyset = +\infty$). We shall write \xrightarrow{S} for the convergence when restricted to the set S . For $\delta \geq 0$, we denote Π_S^δ the (*orthogonal*) δ -*projection mapping* onto S defined as

$$\Pi_S^\delta(x) := \{y \in S : \|x - y\|^2 \leq d_S^2(x) + \delta^2\}.$$

For $\delta = 0$, we simply write $\Pi_S(x) := \Pi_S^0(x)$. It is known that Π_S is nonempty-valued on a dense subset of $H \setminus S$ ([9]).

For an extended real-valued function $\varphi : H \rightarrow \overline{\mathbb{R}}$, we denote $\text{dom}\varphi := \{x \in H \mid \varphi(x) < +\infty\}$ and $\text{epi}\varphi := \{(x, \alpha) \in H \times \mathbb{R} \mid \varphi(x) \leq \alpha\}$. Function φ is lower semi-continuous (lsc) if $\text{epi}\varphi$ is closed. The *contingent directional derivative* of φ at $x \in \text{dom}\varphi$ in the direction $v \in H$ is

$$\varphi'(x, v) := \liminf_{t \rightarrow 0^+, w \rightarrow v} \frac{\varphi(x + tw) - \varphi(x)}{t}.$$

A vector $\xi \in H$ is called a *proximal subgradient* of φ at $x \in H$, written $\xi \in \partial_P \varphi(x)$, if there are $\rho > 0$ and $\sigma \geq 0$ such that

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle - \sigma \|y - x\|^2, \quad \forall y \in B_\rho(x);$$

a *Fréchet subgradient* of φ at x , written $\xi \in \partial_F \varphi(x)$, if

$$\varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle + o(\|y - x\|), \quad \forall y \in H;$$

and a *basic (or Limiting) subdifferential* of φ at x , written $\xi \in \partial_L \varphi(x)$, if there exist sequences $(x_k)_k$ and $(\xi_k)_k$ such that

$$x_k \xrightarrow{\varphi} x, \text{ (i.e., } x_k \rightarrow x \text{ and } \varphi(x_k) \rightarrow \varphi(x)), \xi_k \in \partial_P \varphi(x_k), \xi_k \rightharpoonup \xi.$$

If $x \notin \text{dom} \varphi$, we write $\partial_P \varphi(x) = \partial_F \varphi(x) = \partial_L \varphi(x) = \emptyset$. In particular, if S is a closed set and $s \in S$, we define the *proximal normal cone* to S at s as $N_S^P(s) = \partial_P I_S(s)$, the *Fréchet normal* to S at s as $N_S^F(s) = \partial_F I_S(s)$, the *limiting normal cone* to S at s as $N_S^L(s) = \partial_L I_S(s)$, and the *Clarke normal cone* to S at s as $N_S^C(s) = \overline{\text{co}} N_S^L(s)$. Equivalently, we have that $N_S^P(s) = \text{cone}(\Pi_S^{-1}(s) - s)$, where $\Pi_S^{-1}(s) := \{x \in H \mid s \in \Pi_S(x)\}$. The Bouligand and weak Bouligand tangent cones to S at x are defined as

$$T_S^B(x) := \{v \in H \mid \exists x_k \in S, \exists t_k \rightarrow 0, \text{ st } t_k^{-1}(x_k - x) \rightarrow v \text{ as } k \rightarrow +\infty\}$$

$$T_S^w(x) := \{v \in H \mid \exists x_k \in S, \exists t_k \rightarrow 0, \text{ st } t_k^{-1}(x_k - x) \rightarrow v \text{ as } k \rightarrow +\infty\}, \text{ resp.}$$

We also define the *Clarke subgradient* of φ at x , written $\partial_C \varphi(x)$, as the vectors $\xi \in H$ such that $(\xi, -1) \in N_{\text{epi} \varphi}^C(x, \varphi(x))$, and the *singular subgradient* of φ at x , written $\partial_\infty \varphi(x)$, as the vectors $\xi \in H$ such that $(\xi, 0) \in N_{\text{epi} \varphi}^P(x, \varphi(x))$; in particular, if $\xi \in \partial_\infty \varphi(x)$, then there are sequences $x_k \xrightarrow{\varphi} x$, $\xi_k \in \partial_P \varphi(x_k)$, and $\lambda_k \rightarrow 0^+$ such that $\lambda_k \xi_k \rightarrow \xi$. Observe that $\partial_P \varphi(x) \subset \partial_F \varphi(x) \subset \partial_L \varphi(x) \subset \partial_C \varphi(x)$. For all these concepts and their properties we refer to the book [17].

We shall frequently use the following version of Gronwall's Lemma:

Lemma 2.1 (Gronwall's Lemma; see, e.g., [3]) Let $T > 0$ and $a, b \in L^1(t_0, t_0 + T; \mathbb{R})$ such that $b(t) \geq 0$ a.e. $t \in [t_0, t_0 + T]$. If, for some $0 \leq \alpha < 1$, an absolutely continuous function $w : [t_0, t_0 + T] \rightarrow \mathbb{R}_+$ satisfies

$$(1 - \alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t) \quad \text{a.e. } t \in [t_0, t_0 + T],$$

then

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(t_0) e^{\int_{t_0}^t a(\tau) d\tau} + \int_{t_0}^t e^{\int_s^t a(\tau) d\tau} b(s) ds, \quad \forall t \in [t_0, t_0 + T].$$

2.2 Some examples

Example 2.1 (*Parabolic Variational Inequalities*). Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset with a smooth boundary $\partial\Omega$. Let us consider the following boundary value problem, with Signorini conditions, of finding a function $(t, x) \mapsto u = u(t, x)$ such that

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f, & (t, x) \in [0, T] \times \Omega, \\ u(0, x) = u_0(x), & x \in \Omega \text{ (initial condition)} \\ u \geq 0, \frac{\partial u}{\partial n} \geq 0 \text{ and } u \frac{\partial u}{\partial n} = 0 & \text{for } (t, x) \in [0, T] \times \partial\Omega. \end{cases}$$

It is well-known that the weak formulation of problem (P) is given by the following parabolic variational inequalities

$$(VI) \begin{cases} \text{Find } u \in \mathcal{C} \text{ such that} \\ \int_{\Omega} u'(t)(v(t) - u(t)) dx + \int_{\Omega} \nabla u(t) \cdot \nabla (v(t) - u(t)) dx \geq \\ \int_{\Omega} f(t)(v(t) - u(t)) dx, \quad \forall v \in \mathcal{C}, \text{ a.e. } t \in [0, T]. \end{cases}$$

Here, $\mathcal{C} = \{v \in L^2(0, T; H^1(\Omega)) : v(t) \in C \text{ for a.e. } t \in [0, T]\}$, where $C = \{v \in H^1(\Omega) : v \geq 0 \text{ on } \partial\Omega\}$. It is easy to see that the parabolic variational inequality (VI) is of the form (1). The convexity structure of the set C (since it is a closed convex cone) makes the problem (VI) standard and may be straightforward. Let us consider now a function $g : \mathbb{R} \rightarrow \mathbb{R}$ and define the new set C with the associated set \mathcal{C}

$$C = \{v \in H^1(\Omega) : g(v(x)) \geq 0 \text{ for } x \in \partial\Omega\}.$$

The set C is no more convex and some sufficient conditions on the function g are necessary to ensure the prox-regularity of the sets C and \mathcal{C} (see [2] for more details).

Example 2.2 (Nonlinear Differential Complementarity Systems). Let us consider the following ordinary differential equation, coupled with a complementarity condition,

$$(NDCS) \begin{cases} \dot{x}(t) = f(x(t)) + \lambda(t), & t \in [0, T] \\ 0 \leq \lambda(t) \perp g(x(t)) \geq 0, \end{cases}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is of class C^1 and $\lambda : [0, T] \rightarrow \mathbb{R}^m$ is a Lagrange multiplier (unknown function). We have,

$$0 \leq \lambda(t) \perp g(x(t)) \geq 0 \iff -\lambda(t) \in N_{\mathbb{R}_+^m}(g(x(t))).$$

Hence,

$$\dot{x}(t) \in f(x(t)) - N_{\mathbb{R}_+^m}(g(x(t))).$$

We have, $N_{\mathbb{R}_+^m}(g(x(t))) = \partial I_{\mathbb{R}_+^m}(g(x(t)))$. If we suppose a qualification condition such as, e.g., ∇g is surjective, then, using classical chain rules (see e.g. [21]), we get

$$\partial(I_{\mathbb{R}_+^m} \circ g)(x) = \nabla g(x)^T N_{\mathbb{R}_+^m}(g(x)).$$

By setting $C = \{x \in \mathbb{R}^n : g(x) \geq 0\}$, it is easy to see that problem (NDCS) is equivalent to the following differential inclusion

$$\dot{x}(t) \in f(x(t)) - N_C(x(t)),$$

which is of the form of (1). Under some sufficient conditions on the vectorial function g (see [2, Theorem 3.5]), we show that the set C is r -prox-regular.

Many problems in power converters electronics and unilateral mechanics can be modeled by nonlinear differential complementarity problems of the form (NDCS) (see e.g. [1] and [22]).

3 Differential inclusions involving maximal monotone operators

We review in this section some aspects of the theory of differential inclusions involving maximal monotone operators. Namely, we provide an invariance result for associated closed sets that we use in the sequel.

Given a set-valued operator $A : H \rightrightarrows H$, which we identify with its graph, we denote its *domain* by $\text{dom}A := \{x \in H \mid A(x) \neq \emptyset\}$. Operator A is *monotone* if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \quad \text{for all } (x_1, y_1), (x_2, y_2) \in A,$$

and α -*hypomonotone* for $\alpha \geq 0$ if the operator $A + \alpha \text{id}$ is monotone, where id is the identity mapping. We say that A is *maximal monotone* if A is monotone and coincides with every monotone operator containing its graph. In such a case, it is known that Ax is convex and closed for every $x \in H$. We shall denote by $(Ax)^0$, $x \in \text{dom}A$, the set of minimal norm vectors in Ax ; i.e., $(A(x))^0 := \{y \in A(x) \mid \|y\| = \min_{z \in A(x)} \|z\|\}$; hence, for any vector $x \in \text{dom}A$ and $y \in H$, the set $\Pi_{A(x)}(y)$ is a singleton and we have that $(y - A(x))^0 = y - \Pi_{A(x)}(y)$.

We consider the following differential inclusion

$$\dot{x}(t) \in f(x(t)) - A(x(t)), \quad t \in [0, \infty), \quad x(0; x_0) = x_0 \in \text{dom}A, \quad (3)$$

governed by a maximal monotone operator $A : H \rightrightarrows H$, which is subject to a perturbation by a $(\kappa-)$ Lipschitz continuous mapping $f : H \rightarrow H$. We recall the following result on the existence of solutions of (3); for more details we refer to the book [7].

Proposition 3.1 *System (3) has a (strong) unique absolute continuous solution such that, for all $s, t \geq 0$ and all $x_0, y_0 \in \text{dom}A$,*

$$\begin{cases} x(s; x(t; x_0)) = x(t + s; x_0), \\ \|x(t; x_0) - x(t; y_0)\| \leq e^{\kappa t} \|x_0 - y_0\|, \\ \left\| \frac{d^+ x(t; x_0)}{dt} - [f(x(t; x_0)) - A(x(t; x_0))]^0 \right\| = \|f(x(t; x_0)) - \Pi_{A(x(t; x_0))}(f(x(t; x_0)))\|. \end{cases}$$

Moreover, the function $t \rightarrow \frac{d^+ x(t)}{dt}$ is right-continuous at $t \geq 0$ and we have

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq e^{\kappa t} \left\| \frac{d^+ x(0)}{dt} \right\|. \quad (4)$$

We are going to characterize those closed sets which are invariant with respect to (3).

Definition 3.1 *A closed set $S \subset H$ is invariant for (3) if every solution of (3) starting in S remains in this set for all time $t \geq 0$.*

Due to the semigroup property in Proposition 3.1, it is immediately seen that S is invariant iff every solution of (3) starting in S remains in this set for

all sufficiently small time $t \geq 0$. The issue with these sets, also referred to as *viable sets* for (3); see, [6], is to find good characterizations via explicit criteria, which do not require an a-priori computation of the solution of (3). An extensive research has been done to solve this problem for different kinds of differential inclusions and equations ([9, 10]). Complete primal and dual characterizations are given in [4, 5].

The following result will be useful.

Theorem 3.1 *Suppose S is a closed subset of $\text{dom}A$ and take $x_0 \in S$. Let $x(\cdot)$ be an absolutely continuous function such that $x(0) = x_0$ and $\dot{x}(t) \in f(x(t)) - A(x(t))$ for ae $t \in [0, T]$. If there are real numbers $m, \rho > 0$ such that, for any $x \in S \cap B(x_0, \rho)$,*

$$\sup_{\xi \in N_S^P(x)} \min_{x^* \in A(x) \cap B(\theta, m)} \langle \xi, f(x) - x^* \rangle \leq 0, \quad (5)$$

then there is some $T^ \in (0, T]$ such that $x(t) \in S$ for all $t \in [0, T^*]$.*

We start by recalling the following lemma (see, e.g., [10, 20]), which is a consequence of Ekeland's Variational principle [11].

Lemma 3.1 *Suppose that S is closed. Then, for $x \in H \setminus S$ and $s \in \Pi_S^\delta(x)$, with $\delta > 0$, there exist $s_\delta \in S$ and $x_\delta \in H$ such that*

$$\begin{cases} x_\delta - s_\delta \in N_S^P(s_\delta), \\ \|x_\delta - s_\delta - (x - s)\| \leq 2\delta, \\ \|s - s_\delta\| \leq \delta, \quad \|x - x_\delta\| \leq \delta. \end{cases}$$

In addition, if $x \in B(x_0, \rho)$ for some $x_0 \in S$ and $\rho > 0$, then s_δ also satisfies

$$\|s_\delta - x_0\|^2 \leq 6\rho^2 + 8\delta^2. \quad (6)$$

Proof. (of Theorem 3.1) Fix $\varepsilon > 0$, and let positive reals m, r be as in (5). Since f is Lipschitz there is a constant $M > 0$ such that $f(x) - A(x) \cap B(\theta, m) \subset B(\theta, M)$ for all $x \in K := B(x_0, \rho) \cap S$. We choose positive numbers \bar{t}, δ and positive integer N such that

$$\frac{\bar{t}^2}{N}(M^2 + 4M + 1) < \frac{\varepsilon^2}{4}, \quad M\bar{t} < \min\left\{\frac{\varepsilon N}{2}, \frac{\rho}{2\sqrt{3}}\right\}, \quad \delta < \min\left\{\frac{\bar{t}}{N}, \frac{\rho}{4}\right\}, \quad (7)$$

and denote by $\pi := \{t_0, t_1, \dots, t_N\}$ the uniform partition of the interval $[0, \bar{t}]$; hence, $d(\pi) := \max_{0 \leq i \leq N-1} (t_{i+1} - t_i) = \frac{\bar{t}}{N}$. In view of condition (5), $(f(x_0) - A(x_0)) \cap B(\theta, M) \neq \emptyset$. Then we pick an $s_0^* \in f(x_0) - A(x_0)$ such that $\|s_0^*\| \leq M$ and consider the differential equation

$$\begin{cases} \dot{z}_0(t) = s_0^* \in f(x_0) - A(x_0), \quad t \in [t_0, t_1] \\ z_0(t_0) = x_0; \end{cases}$$

hence, we obviously have that $z_1 := z_0(t_1) = x_0 + t_1 s_0^*$. According to Lemma 3.1, there is (y_1, s_1) such that $y_1 - s_1 \in N_S^P(s_1)$ and $\|s_1 - x_0\|^2 \leq 6\|z_1 - x_0\|^2 + 8\delta^2 \leq 6\|t_1 s_0^*\|^2 + 8\delta^2 \leq 6t_1^2 M^2 + 8\delta^2 < \rho^2$ (recall (6)), together with the other properties listed in Lemma 3.1. Thus, since $y_1 - s_1 \in N_S^P(s_1)$ and $s_1 \in K$, from the current hypothesis (5) we find $s_1^* \in (f(s_1) - A(s_1)) \cap B(\theta, M)$ such that

$$\langle y_1 - s_1, s_1^* \rangle \leq 0.$$

Now, with this vector s_1^* in hand, we consider the following differential equation

$$\begin{cases} \dot{z}_1(t) = s_1^*, & t \in [t_1, t_2] \\ z_1(t_1) = z_0(t_1). \end{cases}$$

By repeating the argument above, for each $i \in \overline{2, N-1}$ we find (y_i, s_i) such that $y_i - s_i \in N_S^P(s_i)$ and $s_i^* \in (f(s_i) - A(s_i)) \cap B(\theta, M)$ such that $s_i \in K$ and

$$\langle y_i - s_i, s_i^* \rangle \leq 0,$$

and consider the corresponding differential equation

$$\begin{cases} \dot{z}_i(t) = s_i^*, & t \in [t_i, t_{i+1}] \\ z_i(t_i) = z_{i-1}(t_i). \end{cases} \quad (8)$$

Next, we define the absolutely continuous trajectory $z(t)$ as

$$z(t) := z_i(t) \quad \text{for } t \in [t_i, t_{i+1}].$$

We claim that

$$d(z(t), S) \leq \varepsilon \quad \text{for all } t \in [0, \bar{t}]. \quad (9)$$

Indeed, for every $i = 1, \dots, N-1$ and $\hat{s}_i \in \Pi_S^\delta(z_i) (= \Pi_S^\delta(z_{i-1}(t_i)))$,

$$\begin{aligned} d_S^2(z_i(t_{i+1})) &\leq \|z_i(t_{i+1}) - \hat{s}_i\|^2 \\ &= \|z_i(t_{i+1}) - z_{i-1}(t_i)\|^2 + \|z_{i-1}(t_i) - \hat{s}_i\|^2 \\ &\quad + 2\langle z_i(t_{i+1}) - z_{i-1}(t_i), z_{i-1}(t_i) - \hat{s}_i \rangle \\ &\leq (t_{i+1} - t_i)^2 M^2 + d_S^2(z_{i-1}(t_i)) + \delta^2 \\ &\quad + 2(t_{i+1} - t_i) \langle s_i^*, z_{i-1}(t_i) - \hat{s}_i - (y_i - s_i) \rangle + 2(t_{i+1} - t_i) \langle s_i^*, y_i - s_i \rangle \\ &\leq (t_{i+1} - t_i)^2 M^2 + d_S^2(z_{i-1}(t_i)) + \delta^2 + 4\delta M(t_{i+1} - t_i) \\ &= d_S^2(z_{i-1}(t_i)) + (t_{i+1} - t_i) d(\pi)(M^2 + 4M + 1). \end{aligned}$$

Then, by summing up over i , and taking into account that

$$d_S^2(z_0(t_1)) \leq \|z_0(t_1) - x_0\|^2 \leq t_1^2 M^2 = t_1 M^2 d(\pi),$$

we obtain

$$\begin{aligned} d_S^2(z_i(t_{i+1})) &\leq d_S^2(z_0(t_1)) + d(\pi)(t_{i+1} - t_1)(M^2 + 4M + 1) \\ &\leq d(\pi)(t_{i+1})(M^2 + 4M + 1) \leq \frac{\varepsilon^2}{4}, \end{aligned}$$

and, so, for $t \in [t_i, t_{i+1}]$

$$\begin{aligned} d_S^2(z(t)) &= d_S^2(z_i(t)) = d_S^2(s_i^*(t - t_i) + z_{i-1}(t_i)) \\ &\leq 2(t - t_i)^2 M^2 + 2d_S^2(z_{i-1}(t_i)) \\ &\leq 2d(\pi)^2 M^2 + \frac{\varepsilon^2}{2} \leq \varepsilon^2, \end{aligned}$$

and (9) follows.

At this step we consider the solution $x(\cdot)$ on $[0, \bar{t}]$. We define the absolutely continuous function $l : [t_i, t_{i+1}] \rightarrow H$ as

$$l(t) := s_i - z(t),$$

which is easily shown to satisfy

$$\dot{z}(t) \in f(z(t) + l(t)) - A(z(t) + l(t)), \quad \text{ae } t \in [t_i, t_{i+1}], \quad z(t_i) = z_{i-1}(t_i).$$

Moreover, invoking (9), for all $t \in [t_i, t_{i+1}]$

$$\begin{aligned} \|l(t)\|^2 &= \|s_i - z(t)\|^2 = \|s_i - z_i(t)\|^2 \\ &\leq 2\|z_{i-1}(t_i) - z_i(t)\|^2 + 2\|s_i - z_{i-1}(t_i)\|^2 \\ &\leq 2(t - t_i)^2 M^2 + 4\|\hat{s}_i - s_i\|^2 + 4\|\hat{s}_i - z_{i-1}(t_i)\|^2 \\ &\leq 2(t - t_i)^2 M^2 + 4d_S^2(z_{i-1}(t_i)) + 8\delta^2 \\ &\leq 2d(\pi)^2 M^2 + \varepsilon^2 + 8\delta^2 < 2\varepsilon^2, \end{aligned}$$

and so

$$\|l(t)\| < 2\varepsilon. \tag{10}$$

Now, using the monotonicity of A , we have that for ae $t \in [t_i, t_{i+1}]$

$$\langle f(z(t) + l(t)) - \dot{z}(t) - f(x(t)) + \dot{x}(t), z(t) + l(t) - x(t) \rangle \geq 0,$$

which, by the Lipschitz condition of f and relation (10), gives us

$$\begin{aligned} \langle \dot{z}(t) - \dot{x}(t), z(t) - x(t) \rangle &\leq 2\varepsilon \|f(z(t) + l(t)) - \dot{z}(t) - f(x(t)) + \dot{x}(t)\| \\ &\quad + \|z(t) - x(t)\| \|f(z(t) + l(t)) - f(x(t))\| \\ &\leq 2\varepsilon \kappa \|z(t) + l(t) - x(t)\| \\ &\quad + 2\varepsilon \|\dot{x}(t) - \dot{z}(t)\| + \kappa \|z(t) - x(t)\| \|z(t) + l(t) - x(t)\| \\ &\leq 4\varepsilon \kappa \|z(t) - x(t)\| + 4\varepsilon^2 \kappa + 2\varepsilon c + \kappa \|z(t) - x(t)\|^2, \end{aligned}$$

where c is any such that $\|\dot{x}(t) - \dot{z}(t)\| \leq c$ for all $t \in [0, \bar{t}]$. By Gronwall's Lemma we get, for very $t \in [0, \bar{t}]$,

$$\|z(t) - x(t)\| \leq \left(\frac{4\varepsilon^2 \kappa + 2\varepsilon c}{\kappa} \right)^{\frac{1}{2}} e^{\kappa t} + 4\varepsilon(e^{\kappa t} - 1),$$

so that

$$d_S(x(t)) \leq d_S(z(t)) + \|z(t) - x(t)\| \leq \left(\frac{4\varepsilon^2 \kappa + 2\varepsilon c}{\kappa} \right)^{\frac{1}{2}} e^{\kappa \bar{t}} + 4\varepsilon e^{\kappa \bar{t}}.$$

The conclusion follows as ε goes to 0. ■

4 The existence result

In this section, we use tools from convex and variational analysis to prove the existence of a solution for the differential inclusion (1),

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)) & \text{ae } t \geq 0 \\ x(0, x_0) = x_0 \in C, \end{cases}$$

where N_C is the proximal, or, equivalently, the limiting, normal cone to an r -uniformly prox-regular closed subset C of H , and f is a Lipschitz continuous mapping. We shall denote by $x(\cdot; x_0)$ the solution of this inclusion.

Definition 4.1 (see [14, 19]) *For positive numbers r and α , a closed set S is said to be (r, α) -prox-regular at $\bar{x} \in S$ provided that one has $x = \Pi_S(x + v)$, for all $x \in S \cap B(\bar{x}, \alpha)$ and all $v \in N_S^P(x)$ such that $\|v\| < r$. The set S is r -prox-regular (resp., prox-regular) at \bar{x} when it is (r, α) -prox-regular at \bar{x} for some real $\alpha > 0$ (resp., for some numbers $r, \alpha > 0$). The set S is said to be r -uniformly prox-regular when $\alpha = +\infty$.*

It is well-known and easy to check that when S is r -uniformly prox-regular, then for every $x \in S$, $N_S^P(x) = N_S^C(x)$; thus, for such sets we will simply write $N_S(x)$ to refer to each one of these cones, and write $T_S(x)$ to refer to the Bouligand tangent cone $T_S^B(x) = (N_S(x))^*$.

We have the following property of r -uniformly prox-regular sets, which can be easily checked.

Proposition 4.1 *Let S be a closed subset of H . If S is r -uniformly prox-regular, then the set-valued mapping defined by $x \mapsto N_S^P(x) \cap \mathbb{B}$ is $\frac{1}{r}$ -hypomonotone.*

Before we state the main theorem of this section we give a useful characterization of prox-regularity.

Lemma 4.1 *The following statements are equivalent for every closed set $C \subset H$ and every $m > 0$,*

- (a) *C is r' -uniformly prox-regular for every $r' < r$,*
- (b) *the mapping $N_C \cap B(\theta, m) + \frac{m}{r} \text{id}$ is monotone,*
- (c) *there exists a maximal monotone operator A defined on H such that*

$$N_C(x) \cap B(\theta, m) + \frac{m}{r}x \subset A(x) \subset N_C(x) + \frac{m}{r}x \quad \text{for every } x \in C.$$

Proof. The equivalence (a) \iff (b) is given in [19, Theorem 4.1], while the implication (c) \implies (b) is immediate. Then we only have to prove that (b) \implies (c). If (b) holds, we choose a maximal monotone operator A , which extends the monotone mapping $N_C \cap B(\theta, m) + \frac{m}{r} \text{id}$, such that $C \subset \text{dom} A \subset \overline{\text{co}} C$ (see, e.g., [7]). Moreover, we have that

$$N_C(x) \cap B(\theta, m) + \frac{m}{r}x \subset A(x) \subset N_C(x) + \frac{m}{r}x, \forall x \in C. \quad (11)$$

Indeed, the first inclusion is obvious. If $x \in C$ and $\xi \in A(x)$, then for any $y \in C$ we have $\frac{m}{r}y \in A(y)$ (since $\theta \in N_C(y) \cap B(\theta, m)$) and, so, $\langle \xi - \frac{m}{r}y, x - y \rangle \geq 0$. This implies

$$\langle \xi - \frac{m}{r}x, y - x \rangle \leq \frac{m}{r}\|y - x\|^2,$$

which proves that $\xi - \frac{m}{r}x \in N_C(x)$, for every $\xi \in A(x)$. Hence, $A(x) \subset N_C(x) + \frac{m}{r}x$. ■

We also need some properties of the solution of (1).

Lemma 4.2 *If $x(\cdot; x_0)$ is a solution of (1), then for ae $t \in [0, T]$ we have*

$$\langle \dot{x}(t), f(x(t)) - \dot{x}(t) \rangle = 0, \quad (12)$$

$$\|f(x(t)) - \dot{x}(t)\| \leq \|f(x(t))\|, \quad (13)$$

$$\|\dot{x}(t)\| \leq \min\{\|f(x(t))\|, \|f(x_0)\|e^{\kappa t}\}, \quad \|x(t) - x_0\| \leq t\|f(x_0)\|e^{\kappa t}. \quad (14)$$

Consequently, $x(\cdot; x_0)$ is the unique solution of (1) on $[0, T]$.

Proof. Let $t \in (0, T]$ be a differentiability point of the solution $x(\cdot)$. Then there is some $\delta > 0$ such that

$$\langle f(x(t)) - \dot{x}(t), x(s) - x(t) \rangle \leq \delta \|x(s) - x(t)\|^2, \quad \text{for all } s \in [0, T],$$

and, so, by dividing on $s - t$ and taking the limit as $s \downarrow t$ we derive that

$$\langle f(x(t)) - \dot{x}(t), \dot{x}(t) \rangle \leq 0.$$

Similarly, when $s \uparrow t$ we get $\langle f(x(t)) - \dot{x}(t), \dot{x}(t) \rangle \geq 0$, which yields (12). Since $f(x(t)) - \dot{x}(t) \in N_C(x(t))$ and $\dot{x}(t) \in T_C^B(x(t))$, statement (12) means that $f(x(t)) - \dot{x}(t) = \Pi_{N_C(x(t))}(f(x(t)))$ and this yields (13), $\|f(x(t)) - \dot{x}(t)\| \leq \|f(x(t))\|$. Moreover, using (12), we have (for ae $t \in [0, T]$)

$$\|\dot{x}(t)\|^2 = \langle \dot{x}(t), \dot{x}(t) \rangle = \langle \dot{x}(t), f(x(t)) \rangle \leq \|\dot{x}(t)\| \|f(x(t))\|, \quad (15)$$

which gives us $\|\dot{x}(t)\| \leq \|f(x(t))\|$. Then

$$\begin{aligned} \frac{d}{dt} \|x(t) - x_0\|^2 &= 2\langle x(t) - x_0, \dot{x}(t) \rangle \leq 2\|x(t) - x_0\| \|f(x(t))\| \\ &\leq 2\|x(t) - x_0\| (\|f(x_0)\| + \kappa\|x(t) - x_0\|) \\ &= 2\|f(x_0)\| \|x(t) - x_0\| + 2\kappa\|x(t) - x_0\|^2, \end{aligned}$$

which by Lemma 2.1 gives us

$$\|x(t) - x_0\| \leq \frac{\|f(x_0)\|}{\kappa} (e^{\kappa t} - 1) \leq \|f(x_0)\| t e^{\kappa t}, \quad (16)$$

so that, using the inequality of the middle together with (15),

$$\begin{aligned} \|\dot{x}(t)\| &\leq \|f(x(t))\| \leq \|f(x_0)\| + \kappa\|x(t) - x_0\| \\ &\leq \|f(x_0)\| + \|f(x_0)\|(e^{\kappa t} - 1) = \|f(x_0)\|e^{\kappa t}. \end{aligned}$$

This proves (13) and (14).

To finish we need to check the uniqueness of the solution. Proceeding by contradiction, we assume that $y(\cdot)$ is another solution on $[0, T]$ of (1). Then for all $t \in [0, T]$ such that $\|f(x(t))\| + \|f(y(t))\| > 0$ and $f(y(t)) - \dot{y}(t) \in N_C(y(t))$ we have

$$\frac{f(y(t)) - \dot{y}(t)}{\|f(x(t))\| + \|f(y(t))\|} \in N_C(y(t)) \cap \mathbb{B},$$

and similarly for $x(\cdot)$. Then, by the r -uniformly prox-regularity hypothesis on C ,

$$\langle \dot{x}(t) - \dot{y}(t), x(t) - y(t) \rangle \leq \left(\kappa + \frac{1}{r} (\|f(x(t))\| + \|f(y(t))\|) \right) \|x(t) - y(t)\|^2; \quad (17)$$

this inequality also holds when $\|f(x(t))\| + \|f(y(t))\| = 0$ as a consequence of (15). By applying Gronwall's Lemma (Lemma 2.1) with the function $\frac{1}{2}\|x(t) - y(t)\|^2$, and observing that $x(0) = y(0) = x_0$, it follows that $x(t) = y(t)$ for every $t \in [0, T]$. ■

The main result is given in the following theorem, using a convex analysis approach, while Theorem 4.2 below provides more properties of the solution, which will be used later on.

Theorem 4.1 *System (1) has a unique solution $x(\cdot, x_0)$ starting at $x_0 \in C$, which is Lipschitz on every bounded interval.*

Proof. We fix a sufficiently large $m > 0$ and choose a $T_0 > 0$ such that

$$\|f(x_0)\| + \kappa(\|f(x_0)\|T_0e^{(\kappa+\frac{m}{r})T_0} + 1) \leq m. \quad (18)$$

By Lemma 4.1(c) we consider a maximal monotone extension A such that, for all $x \in C$,

$$N_C(x) \cap B(\theta, m) + \frac{m}{r}x \subset A(x) \subset N_C(x) + \frac{m}{r}x. \quad (19)$$

According to [7], the differential inclusion

$$\begin{cases} \dot{x}(t) \in f(x(t)) + \frac{m}{r}x(t) - A(x(t)), & \text{ae } t \in [0, T_0] \\ x(0) = x_0 \in C, \end{cases} \quad (20)$$

has a unique solution $x(\cdot)$ such that $x(t) \in \text{dom}A \subset \overline{\text{co}}C$ for all $t \in [0, T_0]$, as well as (see, e.g., [4])

$$\left\| \frac{d^+x(t)}{dt} \right\| \leq e^{(\kappa+\frac{m}{r})t} \left\| \frac{d^+x(0)}{dt} \right\| \leq e^{(\kappa+\frac{m}{r})t} \|\Pi_{A(x_0)}(f(x_0) + \frac{m}{r}x_0)\|.$$

Moreover, since $\frac{m}{r}x_0 \in A(x_0)$ (due to (19)), for all $t \in [0, T_0]$

$$\left\| \frac{d^+x(t)}{dt} \right\| \leq e^{(\kappa + \frac{m}{r})t} \|f(x_0)\| \leq e^{(\kappa + \frac{m}{r})T_0} \|f(x_0)\| =: k,$$

and, hence,

$$\|x(t) - x_0\| \leq kT_0, \quad (21)$$

$$\|f(x(t))\| \leq \|f(x_0)\| + \kappa\|x(t) - x_0\| \leq \|f(x_0)\| + \kappa kT_0;$$

in particular, $x(\cdot)$ is k -Lipschitz on $[0, T_0]$.

Next, we want to show that $x(t) \in C$ for every $t \in [0, T_0]$. For this aim we shall apply Theorem 3.1. Given $y \in C \cap B(x_0, kT_0 + 1)$ and $\xi \in N_C(y)$, we define $z := \Pi_{N_C(y)}(f(y)) \in N_C(y)$ (z is well defined since $N_C(y)$ is closed (and convex)). It is easy to see that

$$\|z\| \leq \|f(y)\| \leq \|f(x_0)\| + \kappa\|y - x_0\| \leq \|f(x_0)\| + \kappa(kT_0 + 1) \leq m.$$

Hence, according to (19), we derive that $y^* := z + \frac{m}{r}y \in N_C(y) \cap B(\theta, m) + \frac{m}{r}y \subset A(y)$, with $\|y^*\| \leq \bar{m} := m(1 + \frac{1}{r}(\|x_0\| + kT_0 + 1))$.

Now, since $f(y) - z \in T_C(y)$ we obtain that $\langle \xi, f(y) - z \rangle \leq 0$, which shows that

$$\inf_{v^* \in Ay \cap B(\theta, \bar{m})} \left\langle \xi, f(y) + \frac{m}{r}y - v^* \right\rangle \leq \langle \xi, f(y) + \frac{m}{r}y - y^* \rangle \leq 0. \quad (22)$$

Consequently, according to Theorem 3.1, there is a positive number $T' \in (0, T_0)$ such that $x(t) \in C$ for every $t \in [0, T']$. Moreover, due to (21), for all $t \in [0, T_0]$ we have that $x(t) \in B(x_0, kT_0 + 1)$ and, so, from the argument above we infer that $x(t) \in C$ for all $t \in [0, T_0]$. Whence, since $x(t) \in C$ for $t \in [0, T_0]$, (19) implies that

$$\begin{aligned} \dot{x}(t) &\in f(x(t)) + \frac{m}{r}x(t) - A(x(t)) \subset f(x(t)) + \frac{m}{r}x(t) - N_C(x(t)) - \frac{m}{r}x(t) \\ &= f(x(t)) - N_C(x(t)); \end{aligned}$$

that is, $x(\cdot)$ is a solution of (1) on $[0, T_0]$.

Now, we set

$$T := \sup \{T' > 0 \text{ such that system (1) has a solution } x(\cdot; x_0) \text{ on } [0, T']\};$$

so, $T > 0$ from the paragraph above. If T is finite, then we take a sequence (T_n) such that $T_n \uparrow T$, and denote $x_n(\cdot; x_0)$ the corresponding solution of (1), which is defined on $[0, T_n]$. Let function $x(\cdot; x_0) : [0, T] \rightarrow H$ be defined as

$$x(t; x_0) = x_n(t) \quad \text{if } t \leq T_n.$$

According to Lemma 4.2 (relation (13)), this function is a well-defined Lipschitz continuous function on $[0, T]$, with Lipschitz constant equal to $\|f(x_0)\|e^{\kappa T}$. Thus, we can extend continuously function $x(\cdot; x_0)$ to $[0, T]$ by setting $x(T) :=$

$\lim_{n \rightarrow \infty} x(T_n)$. Since $x(T) \in C$, from the first paragraph we find a $T_1 > 0$ and a solution of (1) on $[0, T+T_1]$ which coincides with $x(\cdot; x_0)$ on $[0, T]$, contradicting the finiteness of T —this is to say that $T = \infty$. ■

An immediate consequence of (the proof of) Theorem 4.1 is that the solution of differential inclusion (1) satisfies the so-called semi-group property,

$$x(t; x(s; x_0)) = x(t+s; x_0) \text{ for all } t, s \geq 0 \text{ and } x_0 \in C. \quad (23)$$

The following theorem gathers further properties of the solution of (1), that we shall use in the sequel. Relation (24) below on the derivative of the solution reenforces the statement of Lemma 4.2.

Theorem 4.2 *Let $x(\cdot; x_0)$, $x_0 \in C$, be the solution of (1). Then the following statements hold true:*

(a) *For every $t \geq 0$, $x(\cdot; x_0)$ is right-derivable at t with*

$$\begin{aligned} \frac{d^+ x(t)}{dt} &= [f(x(t)) - N_C(x(t))]^0 \\ &= f(x(t)) - \Pi_{N_C(x(t))}(f(x(t))) = \Pi_{T_C(x(t))}(f(x(t))), \end{aligned} \quad (24)$$

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq \min\{\|f(x(t))\|, \|f(x_0)\|e^{\kappa t}\}, \quad (25)$$

$$\left\| \frac{d^+ x(t)}{dt} \right\| \leq \left\| \frac{d^+ x(0)}{dt} \right\| e^{\kappa t + \frac{2\|f(x_0)\|}{\kappa r}} (e^{\kappa t} - 1). \quad (26)$$

(b) *The mapping $t \rightarrow \frac{d^+ x(t)}{dt}$ is right-continuous on $[0, T)$.*

(c) *If $y(\cdot; y_0)$, $y_0 \in C$, is the corresponding solution of (1), then for every $t \geq 0$*

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{\kappa t + \frac{\|f(x_0)\| + \|f(y_0)\|}{\kappa r}} (e^{\kappa t} - 1).$$

Proof. We fix $t \geq 0$ (we may suppose that $t = 0$). From the argument used in the proof of Theorem 4.1 we know that for some $m > \|f(x_0)\| + \kappa$ (κ is the Lipschitz constant of f) there exists a maximal monotone operator A such that $x(\cdot) := x(\cdot; x_0)$ is the solution of the following differential inclusion on some interval $[0, \delta]$, $\delta > 0$,

$$\dot{x}(t) \in f(x(t)) + \frac{m}{r}x(t) - A(x(t)), \quad x(0) = x_0,$$

where r comes from the r -uniform prox-regularity of C . W.l.o.g. we may suppose that $\|f(x(t))\| + \kappa < m$ for all $t \in [0, \delta]$ so that (see Proposition 3.1), for every $t \in [0, \delta]$,

$$\frac{d^+ x(t)}{dt} = [f(x(t)) + \frac{m}{r}x(t) - A(x(t))]^0. \quad (27)$$

Since $f(x(t)) \in B(\theta, m)$ we have that

$$\begin{aligned} (f(x(t)) - N_C(x(t)))^0 &= f(x(t)) - \Pi_{N_C(x(t))}(f(x(t))) \\ &= f(x(t)) - \Pi_{N_C(x(t)) \cap B(\theta, m)}(f(x(t))) \\ &= (f(x(t)) - N_C(x(t)) \cap B(\theta, m))^0, \end{aligned}$$

and, so, due to (27), and the inclusions (19):

$$f(x(t)) - N_C(x(t)) \cap B(\theta, m) \subset f(x(t)) + \frac{m}{r}x(t) - A(x(t)) \subset f(x(t)) - N_C(x(t)),$$

we get the first equality in (24). The other two equalities in (24) easily follow from the definition of the orthogonal projection. Moreover, statement (b) is also a consequence of Proposition 3.1. Thus, (24) follows from Lemma 4.2. Finally, (26) and statement (c) follow easily using relation (17) (and Lemma 2.1). ■

The main idea behind the previous existence theorems, Theorems 4.1 and 4.2, as well as the forthcoming results on Lyapunov stability in the next section, is that differential inclusion (1) is in some sense equivalent to a differential inclusion governed by a (Lipschitz continuous perturbation of a) maximal monotone operator. This fact is highlighted in the following corollary. Recall, by Lemma 4.1(c), that for every $m > 0$ the r -uniformly prox-regularity of the set C yields the existence of a maximal monotone operator A_C such that

$$N_C(x) \cap B(\theta, m) + \frac{m}{r}x \subset A_C(x) \subset N_C(x) + \frac{m}{r}x \quad \text{for every } x \in C. \quad (28)$$

Corollary 4.1 *An absolutely continuous function $x(t)$ is a solution of (101) on $[0, T]$; that is,*

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)), \text{ a.e. } t \in [0, T] \\ x(0) = x_0 \in C, \end{cases}$$

if and only if it is (the unique) solution of the following differential inclusion, for some $m > 0$,

$$(DIM) \quad \begin{cases} \dot{x}(t) \in f(x(t)) + \frac{m}{r}x(t) - A_C(x(t)), \text{ a.e. } t \in [0, T] \\ x(0) = x_0 \in C, \end{cases}$$

where the maximal monotone operator $A_C : H \rightrightarrows H$ is defined in (28).

Proof. According to Theorems 4.1 and 4.2 (namely, (25)), differential inclusion (101) has a unique (absolutely continuous) solution $x(t) := x(t; x_0)$ which satisfies $\|\frac{d^+ x(t)}{dt}\| \leq \|f(x_0)\|e^{\kappa T}$ for a.e. $t \in [0, T]$. Then, we find an $m > 0$ such that

$$\dot{x}(t) \in f(x(t)) - N_C(x(t)) \cap B(0, m),$$

and, so, by the definition of A_C above (see (28)) we conclude that $x(t)$ is also the solution of differential inclusion (DIM).

Conversely, if $x(t)$ is a solution of differential inclusion (DIM) for some $m > 0$, then, as it follows from the proof of Theorem 4.1, we get that $x(t) \in C$ for all $t \in [0, T_0]$ for some $T_0 > 0$. Hence, once again by (28), we conclude that $x(t)$ is also a solution of (101) on $[0, T_0]$. Taking into account Lemma 4.1 we show, also as in the proof of Theorem 4.1, that T_0 can be taken to be T . ■

5 Lyapunov stability analysis

In this section, we give explicit characterizations for lsc a -Lyapunov pairs, Lyapunov functions, and invariant sets associated to differential inclusion (1). Recall that $x(\cdot; x_0)$ (or $x(\cdot)$, when any confusion is excluded) refers to the unique solution of (1), which satisfies $x(0; x_0) = x_0$.

Definition 5.1 *Let functions $V, W : H \rightarrow \overline{\mathbb{R}}$ be lsc, with $W \geq 0$, and let an $a \geq 0$. We say that (V, W) is (or forms) an a -Lyapunov pair for differential inclusion (1) if, for all $x_0 \in C$,*

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0) \text{ for all } t \geq 0. \quad (29)$$

In particular, if $a = 0$, we say that (V, W) is a Lyapunov pair; If, in addition, $W = 0$, then V is said to be a Lyapunov function.

A closed set $S \subset C$ is said to be invariant for (1) if the function δ_S is a Lyapunov function.

Equivalently, using (23), it is not difficult to show that a -Lyapunov pairs are those pairs of functions $V, W : H \rightarrow \overline{\mathbb{R}}$ such that the mapping $t \rightarrow e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau$ is nonincreasing. In other words (see, e.g. [4, Proposition 3.2]), for any $x_0 \in C$, there exists $t > 0$ such that

$$e^{as}V(x(s; x_0)) + \int_0^s W(x(\tau; x_0))d\tau \leq V(x_0) \text{ for all } s \in [0, t]. \quad (30)$$

The failure of *regularity* in our Lyapunov candidate-like pairs is mainly carried out by the function V , since the function W can be always regularized to a Lipschitz continuous function on every bounded subset of H as the following lemma shows (see, e.g., [9]).

Lemma 5.1 *Let V, W and a be as in Definition 5.1. Then there exists a sequence of lsc functions $W_k : H \rightarrow \mathbb{R}$, $k \geq 1$, converging pointwisly to W (for instance, $W_k \nearrow W$) such that W_k is Lipschitz continuous on every bounded subset of H . Consequently, (V, W) forms an a -Lyapunov pair for (1) if and only if each (V, W_k) does.*

Now, we give the main theorem of this section, which characterizes lsc a -Lyapunov pairs associated to differential inclusion (1).

Theorem 5.1 *Let functions $V, W : H \rightarrow \overline{\mathbb{R}}$ be lsc, with $W \geq 0$ and $\text{dom}V \subset C$, $a \geq 0$, and let $x_0 \in \text{dom}V$. If there is $\rho > 0$ such that, for any $x \in B(x_0, \rho)$,*

$$\sup_{\xi \in \partial_P V(x)} \min_{x^* \in N_C(x) \cap B(\theta, \|f(x)\|)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0, \quad (31)$$

then there is some $T^ > 0$ such that*

$$e^{at}V(x(t; x_0)) + \int_0^t W(x(\tau; x_0))d\tau \leq V(x_0), \forall t \in [0, T^*].$$

Consequently, the following statements are equivalent provided that either $\partial \equiv \partial_P$ or $\partial \equiv \partial_F$:

- (i) (V, W) is an a -Lyapunov pair for (1);
- (ii) for every $x \in \text{dom}V$ and $\xi \in \partial V(x)$;

$$\langle \xi, (f(x) - N_C(x))^0 \rangle + aV(x) + W(x) \leq 0;$$

- (iii) for every $x \in \text{dom}V$ and $\xi \in \partial V(x)$;

$$\min_{x^* \in N_C(x) \cap B(\theta, \|f(x)\|)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0;$$

- (iv) for every $x \in \text{dom}V$;

$$V'(x; (f(x) - N_C(x))^0) + aV(x) + W(x) \leq 0;$$

- (v) for every $x \in \text{dom}V$;

$$\inf_{x^* \in N_C(x) \cap B(\theta, \|f(x)\|)} V'(x; f(x) - x^*) + aV(x) + W(x) \leq 0.$$

Moreover, when H is finite-dimensional, all the statements above except (ii) are equivalent when $\partial = \partial_L$.

Proof. Let us start with the first part of the theorem. We choose $T > 0$ such that

$$T\|f(x_0)\|e^{\kappa T} \leq \frac{\rho}{2},$$

and put

$$k := 2 \max\{\|f(x_0)\|e^{\kappa T}, \|f(x_0)\| + \kappa T e^{\kappa T} \|f(x_0)\| + \kappa + 1\};$$

$$m := k + \frac{k}{r}(\|x_0\| + \rho).$$

Thanks to Lemma 5.1 we shall assume in what follows that W is Lipschitz continuous on $B(x_0, \rho)$. As before we denote $x(\cdot)$ the solution of (1) on $[0, T]$ satisfying $x(0) = x_0$. According to Theorem 4.2, for a.e $t \in [0, T]$ we have $\|\dot{x}(t)\| \leq \|f(x(t))\|$ and, due to the κ -Lipschitzianity of f ,

$$2\|f(x(t))\| \leq 2\|f(x_0)\| + 2\kappa\|x(t) - x_0\|$$

$$< 2 \max\{\|f(x_0)\|e^{\kappa T}, \|f(x_0)\| + \kappa T e^{\kappa T} \|f(x_0)\| + \kappa + 1\} = k;$$

that is, $\dot{x}(t) \in f(x(t)) - (N_C(x(t)) \cap B(\theta, k))$. Hence, if $A : H \rightrightarrows H$ is the monotone operator defined as

$$A(x) := \begin{cases} N_C(x) \cap B(\theta, k) + \frac{k}{r}x & \text{if } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

then it is immediately seen that $x(\cdot)$ is also the unique solution of the following differential inclusion,

$$\begin{cases} \dot{x}(t) \in f(x(t)) + \frac{k}{r}x(t) - A(x(t)), & \text{ae } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where r comes from the r -uniform prox-regularity of the set C . We introduce now the monotone operator $\tilde{A} : H \times \mathbb{R}^4 \rightrightarrows H \times \mathbb{R}^4$ and the κ -Lipschitz function $\tilde{f} : H \times \mathbb{R}^4 \rightarrow H \times \mathbb{R}^4$ defined as

$$\tilde{A}(x, \mu) := (Ax, 0_{\mathbb{R}^4}) \text{ and } \tilde{f}(x, \mu) := (f(x) + \frac{k}{r}x, 1, 0, 1, 0),$$

and consider the associated differential inclusion

$$\begin{cases} \dot{y}(t) \in \tilde{f}(y(t)) - \tilde{A}(y(t)), & \text{ae } t \in [0, T] \\ y(0) = (x_0, \mu_0) \in C \times \mathbb{R}^4, \end{cases} \quad (32)$$

whose unique solution is given by $y(t) := (x(t), t, 0, t, 0) + (\theta, \mu_0) \in C \times \mathbb{R}^4$. We define the lsc functions $V_n : H \times \mathbb{R}^3 \rightarrow \overline{\mathbb{R}}$, $n \geq 1$, as

$$V_n(x, \alpha, \beta, \gamma) := e^{a\gamma}V(x) + (\alpha - \beta)g_n(\alpha) + \frac{l}{2}(\alpha - \beta)^2,$$

where g_n is an l -Lipschitz extension of the (Lipschitz) function $W(x(\cdot)) - \frac{1}{n}$ from $[0, T]$ to $[-1, T+1]$; hence,

$$\partial_C g_n(\alpha) \subset B(0, l) \text{ for all } \alpha \in [0, T+1].$$

Observe that $\text{epi}V_n \subset \text{dom}\tilde{A}$ and, for every $(x, \alpha, \beta, \gamma) \in \text{dom}V_n$, we have that $\partial_\infty V_n(x, \alpha, \beta, \gamma) \subset (e^{a\gamma}\partial_\infty V(x), 0, 0, 0)$ and

$$\begin{aligned} \partial_P V_n(x, \alpha, \beta, \gamma) &\subset (e^{a\gamma}\partial_P V(x), g_n(\alpha), -g_n(\alpha), ae^{a\gamma}V(x)) \\ &\quad + (\theta, (\alpha - \beta)\partial_C g_n(\alpha) + l(\alpha - \beta), l(\beta - \alpha), 0). \end{aligned} \quad (33)$$

At this step, we pick $t \in [0, T]$ and fix $n \geq 1$ such that $\frac{1}{n} \leq \frac{\rho}{2}$. We denote $y_0 := (x(t), t, t, 0, V_n(x(t), t, t, 0))$, and choose $\varepsilon > 0$ such that (recall that W is Lipschitz continuous on $B(x_0, \rho)$)

$$g_n(\alpha') + 2l|\alpha' - \beta'| - e^{a\gamma'}W(x') \leq \frac{-1}{2n}, \quad (34)$$

for any $(y', \mu') := (x', \alpha', \beta', \gamma', \mu') \in U_\varepsilon := B(y_0, \varepsilon) \cap \text{epi}V_n$. Take vectors $(y, \mu) := (x, \alpha, \beta, \gamma, \mu) \in U_\varepsilon$ and $\xi \in N_{\text{epi}V_n}^P(y, \mu)$. Then $x \in \text{dom}V \subset C$ and (recall Lemma 4.2 (relation (14)))

$$\|x - x_0\| \leq \|x - x(t)\| + \|x(t) - x_0\| \leq \varepsilon + \|f(x_0)\|Te^{\kappa T} < \rho;$$

hence, $x \in \text{dom}V \cap \text{int}(B(x_0, \rho))$ and

$$\|f(x)\| \leq \|f(x_0)\| + \kappa\|x - x_0\| < \|f(x_0)\| + \kappa(1 + \|f(x_0)\|Te^{\kappa T}) \leq \frac{k}{2};$$

thus,

$$f(x) - (N_C(x) \cap B(\theta, \|f(x)\|)) \subset f(x) - (N_C(x) \cap B(\theta, k)) = f(x) + \frac{k}{r}x - A(x). \quad (35)$$

Also, we have that $\xi \in N_{\text{epi}V_n}^P(y, V_n(y))$ (see, e.g., [9, Exercise 2.1]) and, so, due to relation (33), either

$$\xi = \iota(e^{a\gamma}\xi_1, g_n(\alpha) + (\alpha - \beta)(\varsigma + l), -g_n(\alpha) + l(\beta - \alpha), ae^{a\gamma}V(x), -1), \quad (36)$$

for some $\xi_1 \in \partial_P V(x)$, $\varsigma \in \partial_C g_n(\alpha)$ and $\iota \geq 0$, or

$$\xi = \iota(e^{a\gamma}\xi_2, 0) \quad (37)$$

with $\xi_2 \in \partial_\infty V(x)$.

If (36) holds, by the current assumption, there exists an $x^* \in N_C(x) \cap B(\theta, \|f(x)\|)$ such that

$$\langle \xi_1, f(x) - x^* \rangle + aV(x) + W(x) \leq 0.$$

Hence, due to (35), $y^* := (x^* + \frac{k}{r}x, 0_{\mathbb{R}^4})$ belongs to $\tilde{A}(y, \mu) \cap B(\theta, m)$ (since $\|x^* + \frac{k}{r}x\| \leq \|x^*\| + \frac{k}{r}\|x\| \leq \frac{k}{2} + \frac{k}{r}(\|x_0\| + \rho) \leq m$), and satisfies

$$\begin{aligned} \langle \xi, \tilde{f}(y, \mu) - y^* \rangle &= \iota(\langle e^{a\gamma}\xi_1, f(x) - x^* \rangle + g_n(\alpha) + (\alpha - \beta)(\varsigma + l) + ae^{a\gamma}V(x)) \\ &= \iota e^{a\gamma}(\langle \xi_1, f(x) - x^* \rangle + aV(x) + W(x)) \\ &\quad + \iota(g_n(\alpha) + (\alpha - \beta)(\varsigma + l) - e^{a\gamma}W(x)) \\ &\leq \iota(g_n(\alpha) + 2l|\alpha - \beta| - e^{a\gamma}W(x)) \\ &\leq -\frac{\iota}{2n} \leq 0, \end{aligned}$$

where in the last inequality we used (34).

We now consider the case when ξ satisfies (37). Let sequences $x_k \xrightarrow{V} x$, $\nu_k \in \partial_P V(x_k)$ and $\alpha_k \rightarrow 0^+$ be such that $\alpha_k \nu_k \rightarrow \xi_2$ (see [17, Lemma 2.37]). Since $x \in \text{int}B(x_0, \rho)$, we may assume that $x_k \in B(x_0, \rho) \cap \text{dom}V$ for all $k = 1, 2, \dots$. By hypothesis, for every k , there exists $x_k^* \in N_C(x_k) \cap B(\theta, \|f(x_k)\|)$ such that

$$\langle \nu_k, f(x_k) - x_k^* \rangle + aV(x_k) + W(x_k) \leq 0.$$

Since f is Lipschitz, the sequence (x_k^*) is bounded and we may suppose (w.l.o.g.) that it weakly converges to some $x^* \in N_C(x) \cap B(\theta, \|f(x)\|)$, due to the r -uniform prox-regularity of C . Consequently, by multiplying the last inequality above by α_k and next passing to the limit on k we obtain $\langle \xi_2, f(x) - x^* \rangle \leq 0$. Hence, the vector $z^* := (x^* + \frac{k}{r}x, 0_{\mathbb{R}^4})$ belongs to $\tilde{A}(y, \mu) \cap B(\theta, m)$ and satisfies

$$\langle \xi, \tilde{f}(y, \mu) - z^* \rangle = \kappa \langle (e^{a\gamma} \xi_2, 0, 0, 0), (f(x) - x^*, 1, 0, 1, 0) \rangle = \kappa e^{a\gamma} \langle \xi_2, f(x) - x^* \rangle \leq 0.$$

Consequently, according to Theorem 3.1, there is a $T_0 > 0$ such that the solution of (32) on $[0, T_0]$ starting at $(x(t), t, t, 0, V(x(t)))$, which is given by $y(s) = (x(s+t), s+t, t, s, V(x(t)))$, lies in $\text{epi}V_n$; that is, for every $s \in [0, T_0]$,

$$V_n(x(s+t), s+t, t, s) = e^{as}V(x(t+s)) + sg_n(s+t) + \frac{l}{2}s^2 \leq V(x(t)),$$

implying that,

$$e^{as}V(x(t+s)) + \int_0^s g_n(t+\tau)d\tau \leq V(x(t)). \quad (38)$$

We claim that

$$e^{as}V(x(t+s)) + \int_0^s g_n(t+\tau)d\tau \leq V(x(t)) + \frac{2}{n}e^{\max\{1,a\}s}, \quad \forall s \in [0, T-t]. \quad (39)$$

Indeed, if

$$T^* := \sup\{T' > 0, \text{ such that (39) holds on } [0, T']\},$$

then from (38) we have that $T^* \geq T_0 > 0$, while the lsc of V and the continuity of g_n yield that (39) also holds at T^* . If $T^* < T-t$, by (38) there exists $\delta > 0$ such that for all $s \in [0, \delta]$

$$e^{as}V(x(t+T^*+s)) + \int_0^s g_n(t+T^*+\tau)d\tau \leq V(x(t+T^*)).$$

Therefore, for all $s \in [0, \delta]$ we obtain

$$\begin{aligned} & e^{a(T^*+s)}V(x(t+T^*+s)) + \int_0^{T^*+s} g_n(t+\tau)d\tau \\ & \leq e^{aT^*} \left(V(x(t+T^*)) - \int_0^s g_n(t+T^*+\tau)d\tau \right) + \int_0^{T^*} g_n(t+\tau)d\tau \\ & \quad + \int_0^s g_n(t+T^*+\tau)d\tau \\ & \leq V(x(t)) + \frac{2}{n}e^{\max\{1,a\}T^*} + (1 - e^{aT^*}) \int_0^s g_n(t+T^*+\tau)d\tau \\ & \leq V(x(t)) + \frac{2}{n}e^{\max\{1,a\}T^*} + \frac{2\delta(e^{aT^*} - 1)}{n} \leq V(x(t)) + \frac{2}{n}e^{\max\{1,a\}(T^*+s)}, \end{aligned}$$

where in the last inequality we used the fact that $1 + \lambda \leq e^\lambda$ for all $\lambda > 0$. This contradicts the definition of T^* , and so (39) holds true; that is (evaluating at $t = 0$), for all $s \in [0, T]$

$$e^{as}V(x(s)) + \int_0^s W(x(\tau))d\tau - \frac{s}{n} = e^{as}V(x(s)) + \int_0^s g_n(\tau)d\tau \leq V(x_0) + \frac{2}{n}e^{\max\{1,a\}s};$$

hence, as n goes to ∞ , we get $e^{as}V(x(s)) + \int_0^s W(x(\tau))d\tau \leq V(x_0)$ for all $s \in [0, T]$, so, we complete the first part of the theorem.

We turn now to the second part of the theorem. These implications $(iv) \Rightarrow (v)$ and $(ii) \Rightarrow (iii)$ follow from the relation $(f(x) - N_C(x))^0 = f(x) - \Pi_{N_C(x)}(f(x))$, $x \in C$, and the fact that $\|\Pi_{N_C(x)}(f(x))\| \leq \|f(x)\|$.

$(i) \Rightarrow (iv)$. Assuming that (V, W) is an a -Lyapunov pair, for any $x_0 \in \text{dom}V$ and $t > 0$ the solution $x(\cdot) = x(\cdot; x_0)$ satisfies

$$0 \geq t^{-1}(V(x(t)) - V(x_0)) + t^{-1}(e^{at} - 1)V(x(t)) + \int_0^t t^{-1}W(x(\tau))d\tau. \quad (40)$$

Thus, observing that $\frac{x(t) - x_0}{t} \rightarrow [f(x_0) - N_C(x_0)]^0$ (recall Theorem 4.2(a)), and using the lsc of V and W , as $t \downarrow 0$ in the last inequality we get

$$\begin{aligned} V'(x_0, (f(x_0) - N_C(x_0))^0) &= \liminf_{\substack{w \rightarrow (f(x_0) - N_C(x_0))^0 \\ t \downarrow 0}} \frac{V(x_0 + tw) - V(x_0)}{t} \\ &\leq \liminf_{t \downarrow 0} t^{-1}(V(x(t)) - V(x_0)) \leq -aV(x_0) - W(x_0). \end{aligned}$$

$(iv) \Rightarrow (ii)$ and $(v) \Rightarrow (iii)$, when $\partial = \partial_F$ or $\partial = \partial_P$. These implications follow due to the relation $\langle \xi, v \rangle \leq V'(x, v)$ for all $\xi \in \partial_F V(x)$, $x \in \text{dom}V$, and $v \in H$.

$(iii) \Rightarrow (i)$ is an immediate consequence of the first part of the theorem together with (30).

Finally, to prove the last statement of the theorem when $\partial = \partial_L$ in the finite-dimensional case, we first check that $(i) \implies (iii)$. Assume that (i) holds and take $x \in \text{dom}V$ together with $\xi \in \partial_L V(x)$, and let sequences $x_k \xrightarrow{V} x$ together with $\xi_k \in \partial_P V(x_k)$ such that $\xi_k \rightarrow \xi$. Since (iii) holds for $\partial = \partial_P$, for each k there exists $x_k^* \in N_C(x_k) \cap B(\theta, \|f(x_k)\|)$ such that

$$\langle \xi_k, f(x_k) - x_k^* \rangle + aV(x_k) + W(x_k) \leq 0.$$

We may assume that (x_k^*) converges to some $x^* \in N_C(x) \cap B(\theta, \|f(x)\|)$ (thanks to the r -uniform prox-regularity of C), which then satisfies $\langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0$ (using the lsc of the involved functions), showing that (iii) holds. Thus, since (iii) (with $\partial = \partial_P$) $\implies (i)$, we deduce that $(i) \iff (iii)$. This suffices to get the conclusion of the theorem. ■

Because the solution $x(\cdot)$ of differential inclusion (1) naturally lives in C , it is immediate that a (lsc) function $V : H \rightarrow \overline{\mathbb{R}}$ is Lyapunov for (1) iff the function

$V + I_C$ is Lyapunov. Hence, Theorem 5.1 also provides the characterization of Lyapunov functions without any restriction on their domains; for instance, accordingly to Theorem 5.1(iii), V is Lyapunov for (1) iff for every $x \in \text{dom}V \cap C$ and $\xi \in \partial(V + I_C)(x)$ it holds

$$\min_{x^* \in N_C(x) \cap B(\theta, \|f(x)\|)} \langle \xi, f(x) - x^* \rangle + aV(x) + W(x) \leq 0.$$

The point here is that this condition is not completely written by means exclusively of the subdifferential of V . Nevertheless, this condition becomes more explicit in each time one can decompose the subdifferential set $\partial(V + I_C)(x)$. For instance, this is the case, if V is locally Lipschitz and lower regular (particularly convex, see [17, Definition 1.91]). This fact is considered in Corollary 5.1 below. However, the following example shows that we can not get rid of the condition $\text{dom}V \subset C$, in general.

Remark 5.1 *We consider the differential inclusion (1) in \mathbb{R}^2 , with $C := \mathbb{B}$ and $f(x, y) = (-y, x)$, whose unique solution such that $x(0) = (1, 0)$ is $x(t) = (\cos t, \sin t)$. We take $V = I_S$, where*

$$S := \{(1, y) : y \in [0, 1]\},$$

so that $\text{dom}V \cap C = \{(1, 0)\}$. For $\bar{x} := (1, 0)$ and $\xi := (x, y) \in \partial_P V(\bar{x}) = \{(x, y) \mid y \leq 0\}$ we have that

$$\min_{x^* \in N_C(\bar{x}) \cap B(\theta, \|f(\bar{x})\|)} \langle \xi, f(\bar{x}) - x^* \rangle \leq \langle \xi, f(\bar{x}) \rangle = \langle (x, y), (0, 1) \rangle = y \leq 0,$$

which shows that condition (iii) of Theorem 5.1 holds. However, it is clear that V is not a Lyapunov function of (1).

Corollary 5.1 *Let V , W and a be as in Theorem 5.1. Then the following assertions hold:*

(i) *If V is Fréchet differentiable on $\text{dom}V \cap C$, then (V, W) is an a -Lyapunov pair for differential inclusion (1) iff for every $x \in \text{dom}V \cap C$*

$$\langle \nabla V(x), (f(x) - N_C(x))^0 \rangle + aV(x) + W(x) \leq 0.$$

(ii) *If V is locally Lipschitz on $\text{dom}V \cap C$, then (V, W) is an a -Lyapunov pair for differential inclusion (1) if for every $x \in \text{dom}V \cap C$*

$$\langle \xi, (f(x) - N_C(x))^0 \rangle + aV(x) + W(x) \leq 0, \quad \forall \xi \in \partial_L V(x).$$

(iii) *If H is of finite dimension and V is regular and locally Lipschitz on $\text{dom}V \cap C$, then (V, W) is an a -Lyapunov pair for differential inclusion (1) iff for every $x \in \text{dom}V \cap C$,*

$$\langle \xi, (f(x) - N_C(x))^0 \rangle + aV(x) + W(x) \leq 0, \quad \forall \xi \in \partial_L V(x).$$

Proof. (i). Since $x(t) \in C$ for every $t \geq 0$, we have that (V, W) forms an a -Lyapunov pair for (1) iff the pair $(V + I_C, W)$ does. Thus, since $\partial_F(V + I_C)(x) = \nabla V(x) + N_C(x)$ for every $x \in \text{dom}V \cap C$, according to Proposition 1.107 in [17], Theorem 5.1 ensures that (V, W) is an a -Lyapunov pair of (1) iff for every $x \in \text{dom}V \cap C$ and $\xi \in N_C(x)$

$$\langle \nabla V(x) + \xi, (f(x) - N_C(x))^0 \rangle + aV(x) + W(x) \leq 0. \quad (41)$$

Because $\theta \in N_C(x)$ and $(f(x) - N_C(x))^0 \in T_C^B(x) = (N_C(x))^*$, it follow that this last inequality is equivalent to $\langle \nabla V(x), (f(x) - N_C(x))^0 \rangle + aV(x) + W(x) \leq 0$.

(ii). Under the current assumption, for every $x \in V \cap C$ we have that $\partial_L(V + I_C)(x) \subset \partial_L V(x) + N_C(x)$, and we argue as in the proof of statement (i).

(iii). In this case, we argue as above but using the relation $\partial_L(V + I_C)(x) = \partial_L V(x) + N_C(x)$. ■

It the result below, Theorem 5.1 is rewritten in order to characterize invariant sets associated to differential inclusion (1) (see Definition 3.1).

Theorem 5.2 *Given a closed set $S \subset C$ we denote by N_S either N_S^P or N_S^F , and by T_S either T_S^B , T_S^w , $\overline{\text{co}}T_S^w$, or $(N_S)^*$. Then S is an invariant set for (1) iff one of the following equivalent statements hold:*

- (i) $(f(x) - N_C(x))^0 \in T_S(x) \forall x \in S$;
- (ii) $[f(x) - N_C(x)] \cap T_S(x) \cap B(\theta, \|f(x)\|) \neq \emptyset \forall x \in S$;
- (iii) $\inf_{x^* \in [f(x) - N_C(x)] \cap B(\theta, \|f(x)\|)} \langle \xi, x^* \rangle \leq 0, \forall x \in S, \forall \xi \in N_S(x)$.

Proof. Under the invariance of S we write (recall Theorem 4.2)

$$(f(x) - N_C(x))^0 = \frac{d^+x(0; x)}{dt} = \lim_{t \searrow 0} \frac{x(t) - x}{t} \in T_S^B(x),$$

showing that (i) with $T_S(x) = T_S^B(x)$ holds. The rest of the implications follows by applying Theorem 5.1 with the use of the following equalities

$$T_S^B(x) \subset T_S^w(x) \subset \overline{\text{co}}T_S^w(x) \subset (N_S^F(x))^* \subset (N_S^P(x))^*, \quad x \in S,$$

where the star in the superscript refers to the dual cone. ■

6 Stability and observer designs

In this section, we give an application of the results developed in the previous sections, to study the stability and observer design for Lur'e systems involving nonmonotone set-valued nonlinearities. The state of the system is constrained to evolve inside a time-independent prox-regular set. More precisely, let us consider the following problem

$$\dot{x}(t) = Ax(t) + Bu(t), \text{ a.e. } t \in [0, \infty), \quad (42a)$$

$$y(t) = Dx(t), \quad \forall t \geq 0, \quad (42b)$$

$$u(t) \in -N_S(y(t)) \quad \forall t \geq 0, \quad (42c)$$

$$x(0) = x_0 \in D^{-1}(S); \quad (42d)$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $D \in \mathbb{R}^{l \times n}$, $l \leq n$, and $S \subset \mathbb{R}^l$ is a uniformly-prox-regular set.

Using (42b) and (42c), and putting the resulting equation in (42a), we get the following differential inclusion

$$\dot{x}(t) \in Ax(t) - BN_S(Dx(t)), \text{ a.e. } t \in [0, \infty), x(0) = x_0 \in D^{-1}(S). \quad (43)$$

It is well-known that if $D : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear mapping and S is a convex subset of \mathbb{R}^m , then the set

$$D^{-1}(S) := \{x \in \mathbb{R}^n : D(x) \in S\}$$

is always convex. This fails when S is prox-regular (see Example 2 in [2] for a counterexample). The following lemma provides a sufficient condition to ensure that $D^{-1}(S)$ is still prox-regular.

Lemma 6.1 ([23]) *Consider a nonempty, closed, r -prox-regular set S such that S is contained in the range space of a linear mapping $D : \mathbb{R}^n \rightarrow \mathbb{R}^l$. Then the set $D^{-1}(S)$ is r' -uniformly prox-regular with $r' := \frac{r\delta_D^+}{\|D\|^2}$, where δ_D^+ denote the least positive singular value of the matrix D .*

The following proposition shows that system (42), or equivalently (43), can be transformed into a differential inclusion of the form (1).

Proposition 6.1 *Let us consider system (42). Assume that S is contained in the range space of D and there exists a symmetric positive definite matrix P such that $PB = D^T$. Then every solution of (42) is also a solution of the following system*

$$\dot{z}(t) \in f(z(t)) - N_{S'}(z(t)), \text{ a.e. } t \geq 0, z(0) \in S',$$

with $z(t) = P^{\frac{1}{2}}x(t)$, $f = P^{\frac{1}{2}}AP^{-\frac{1}{2}}$ and $S' = (DP^{-\frac{1}{2}})^{-1}(S)$.

Proof. We set $R := P^{\frac{1}{2}}$. According to Lemma 6.1, the set S' is r' -uniformly prox-regular with $r' := \frac{r\delta_D^+}{\|DR^{-1}\|^2}$. Combining this with the basic chain rule (see Theorem 10.6, [21]), for any $x \in \mathbb{R}^n$, one has

$$\begin{aligned} (DR^{-1})^T N_S(DR^{-1}x) &= (DR^{-1})^T \partial I_S(DR^{-1}x) \subset \partial(I_S \circ (DR^{-1}))(x) \\ &= \partial I_{S'}(x) = N_{S'}(x). \end{aligned}$$

By the hypothesis $PB = C^T$, we deduce that $DR^{-1} = (RB)^T$. From the above inclusion, it is easy to see that for a.e. $t \geq 0$, one has

$$\begin{aligned} \dot{z}(t) &\in RAR^{-1}z(t) - RBN_S(DR^{-1}z(t)) \\ &= RAR^{-1}z(t) - (DR^{-1})^T N_S(DR^{-1}z(t)) \subset RAR^{-1}z(t) - N_{S'}(z(t)). \end{aligned} \quad (44)$$

The proof of Proposition 6.1 is thereby completed. ■

The above Proposition proves that under some assumptions, system (42) can be studied within the framework of (1). Let us now investigate the asymptotic stability of differential inclusion (1)

$$\begin{cases} \dot{x}(t) \in f(x(t)) - N_C(x(t)) & \text{ae } t \geq 0 \\ x(0; x_0) = x_0 \in C, \end{cases}$$

at the equilibrium point θ , with the assumption $\theta \in C$ and $f(\theta) = \theta$. Recall that the set C is an r -uniformly prox-regular set ($r > 0$), and that f is a Lipschitz continuous mapping with Lipschitz constant L .

We have the following result which provides a partial extension of [23, Theorem 3.2] (here, we are considering the case where the set C is time-independant).

Theorem 6.1 *Assume that for some $\varepsilon, \delta > 0$, the following holds*

$$\langle x, f(x) \rangle + \delta \|x\|^2 \leq 0 \quad \forall x \in C \cap B(\theta, \varepsilon). \quad (45)$$

Then

$$\lim_{t \rightarrow \infty} x(t, x_0) = \theta \quad \text{for all } x_0 \in \text{int}(B(\theta, \min\{r\delta L^{-1}, \varepsilon\})) \cap C.$$

Proof. We shall verify that the (lsc proper) function $V : H \rightarrow \mathbb{R} \cup \{+\infty\}$, defined by $V(x) := \frac{1}{2}\|x\|^2 + I_C(x)$, satisfies the assumption of Theorem 5.1 (when $W \equiv 0$ and $a = \delta$). We fix $\eta \in (0, \min\{r\delta L^{-1}, \varepsilon\})$, $x \in B(\theta, \eta) \cap C$ and $\xi \in \partial_P V(x) \subset x + N_C(x)$ ([10, Ch. 1, Proposition 2.11]); hence, since $(f(x) - N_C(x))^0 = \Pi_{T_C(x)}(f(x)) \in T_C(x)$ we obtain

$$\langle \xi, (f(x) - N_C(x))^0 \rangle \leq \langle x, (f(x) - N_C(x))^0 \rangle = \langle x, f(x) - \Pi_{N_C(x)}(f(x)) \rangle,$$

so that, by (45),

$$\langle \xi, (f(x) - N_C(x))^0 \rangle \leq -\langle x, \Pi_{N_C(x)}(f(x)) \rangle - 2\delta V(x). \quad (46)$$

Moreover, because $\Pi_{N_C(x)}(f(x)) \in N_C(x)$ and $\theta \in C$, from the r -uniformly prox-regularity of the set C we have

$$\langle \Pi_{N_C(x)}(f(x)), -x \rangle \leq \frac{\|\Pi_{N_C(x)}(f(x))\|}{r} V(x) \leq \frac{\|f(x)\|}{r} V(x),$$

and we get, using (46),

$$\begin{aligned} \langle \xi, f(x) - \Pi_{N_C(x)}(f(x)) \rangle + \delta V(x) &= \langle \xi, (f(x) - N_C(x))^0 \rangle + \delta V(x) \\ &\leq (r^{-1} \|f(x)\| - \delta) V(x). \end{aligned}$$

But, by the choice of η we have $\|f(x)\| = \|f(x) - f(\theta)\| \leq L \|x\| \leq L\eta \leq r\delta$, and so,

$$\langle \xi, f(x) - \Pi_{N_C(x)}(f(x)) \rangle + \delta V(x) \leq 0.$$

Consequently, observing that $\Pi_{N_C(x)}(f(x)) \in N_C(x) \cap B(\theta, \|f(x)\|)$, by Theorem 5.1 we deduce that for every $x_0 \in C \cap \text{int}(B(\theta, \eta))$, there exists $t_0 > 0$ such that

$$e^{\delta t} V(x(t; x_0)) \leq V(x_0) \quad \forall t \in [0, t_0];$$

hence, in particular, $\frac{1}{2}\|x(t; x_0)\|^2 \leq \frac{1}{2}\|x_0\|^2$ and $x(t_0; x_0) \in C \cap \text{int}(B(\theta, \eta))$. This proves that

$$\hat{t}_0 := \sup \{t > 0 \mid e^{\delta t} V(x(s; x_0)) \leq V(x_0) \quad \forall s \in [0, t]\} = +\infty,$$

and we conclude that

$$e^{\delta t} V(x(t; x_0)) \leq V(x_0) \quad \forall t \geq 0,$$

which leads us to the desired conclusion. ■

Corollary 6.1 *Let us consider system (42). Assume that S is uniformly prox-regular set such that S is contained in the rank of D . If there exists a symmetric positive definite matrix P and $\delta > 0$ such that*

$$A^T P + P A \leq -\delta P, \quad P B = D^T, \quad (47)$$

then

$$\lim_{t \rightarrow \infty} x(t; x_0) = \theta \quad \text{for all } x_0 \in \text{int}[(B(\theta, \rho)] \cap S,$$

where $\rho := (2\|R^{-1}\| \|DR^{-1}\| \|RAR^{-1}\|)^{-1} \delta r \delta_{DR^{-1}}^+$.

Proof. Firstly we will show that for any $x \in \mathbb{R}^n$, one has

$$\langle RAR^{-1}x, x \rangle + \frac{\delta}{2}\|x\|^2 \leq 0.$$

Indeed, by the first inequality of (47), for every $x \in \mathbb{R}^n$, one has

$$\langle (A^T P + P A + \theta P)x, x \rangle = \langle (A^T R^2 + R^2 A + \delta R^2)x, x \rangle \leq 0.$$

Since R is positive definite, for any $z = R^{-1}x$, one has

$$\begin{aligned} 0 &\geq \langle (A^T P + P A + \delta P)R^{-1}x, R^{-1}x \rangle = \langle (A^T R + P A R^{-1} + \delta R)x, R^{-1}x \rangle \\ &= \langle (R^{-T} A^T R + R A R^{-1} + \delta I_n)x, x \rangle = 2\langle R A R^{-1}x, x \rangle + \delta\|x\|^2. \end{aligned} \quad (48)$$

Applying Theorem 6.1 to system (44) with $f = R A R^{-1}$, $C = S'$, $r = r'$, we get

$$\lim_{t \rightarrow \infty} z(t; z_0) = \theta,$$

for every $z_0 \in \text{int}[B(\theta, \frac{1}{2}\|R^{-1}AR\|^{-1}r'\delta)] \cap S'$. Combining this with the fact that $x(t) = R^{-1}z(t)$, the conclusion of Corollary 6.1 follows because

$$z = Rx \in \text{int}[B(\theta, \frac{1}{2}\|R^{-1}AR\|^{-1}r'\delta)],$$

for any $x \in \text{int}[B(\theta, \rho)]$. ■

Next let us remind Luenberger-like observers associated to differential inclusion (42). Given $x_0 \in D^{-1}(S)$, we assume that the output equation associated with differential inclusion (42) is

$$y(t) = G(x(t; x_0))$$

where $G \in \mathbb{R}^{p \times n}$ with $p \leq n$.

The Luenberger-like observers associated to differential inclusion (42) has the following form

$$\dot{\hat{x}}(t) = (A - LG)\hat{x}(t) + Ly(t) + B\hat{u}(t), \quad (49a)$$

$$\hat{y}(t) = D\hat{x}(t), \quad (49b)$$

$$\hat{u}(t) \in -N_S(\hat{y}(t)), \quad (49c)$$

$$\hat{x}(0) = z_0, \quad (49d)$$

where $L \in \mathbb{R}^{n \times p}$ is the observer gain. This differential inclusion always has a unique solution, denoted by $\hat{x}(\cdot; z_0)$. We want to find the gain L for the basic observer such that

$$\lim_{t \rightarrow \infty} \|\hat{x}(t; z_0) - x(t; x_0)\| = 0, \text{ for all } z_0 \in B(x_0, \rho) \cap D^{-1}(S) \text{ for some } \rho > 0. \quad (50)$$

We see that if $\hat{x}(\cdot) := \hat{x}(\cdot; z_0)$ is the solution of (49), then it is also the solution of the differential inclusion

$$\dot{\hat{x}}(t) \in (A - LG)\hat{x}(t) + Ly(t) - BN_S(D\hat{x}(t)), \text{ ae } t \geq 0, \hat{x}(0) = z_0. \quad (51)$$

Under the hypothesis

$$\exists P \text{ symmetric positive definite, such that } PB = D^T, \quad (52)$$

similarly to the proof of Proposition 6.1, we have

$$\dot{\hat{z}}(t) \in (RAR^{-1} - RLG')\hat{z}(t) + RLG'z(t) - I_{S'}(\hat{z}(t)),$$

where $G' := GR^{-1}$, $\hat{z}(t) := R\hat{x}(t; z_0)$ and $z(t) = Rx(t; x_0)$, $S' = (DR^{-1})^{-1}(S)$.

On the other hand, one has

$$\|R\|^{-1} \|\hat{z}(t) - z(t)\| \leq \|\hat{x}(t) - x(t)\| \leq \|R^{-1}\| \|\hat{z}(t) - z(t)\|,$$

which means that $\|\hat{z}(t) - z(t)\| \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\|\hat{x}(t) - x(t)\|$ does.

Next, we investigate a general Luenberger-like observers associated to our differential inclusion (1). Following the same idea as above, we assume that $x_0 \in C$ and the output equation associated with differential inclusion (1) is

$$y(t) = \mathcal{G}(x(t; x_0)),$$

where $\mathcal{G} : H \rightarrow H$ is a Lipschitz mapping. We want to find a Lipschitz mapping $\mathcal{L} : H \rightarrow H$ such that the solution $\hat{x}(\cdot; z_0)$ of the differential inclusion

$$\begin{cases} \dot{\hat{x}}(t) \in f(\hat{x}(t)) - \mathcal{L}(\mathcal{G}(\hat{x}(t))) + \mathcal{L}(\mathcal{G}(y(t))) - N_C(\hat{x}(t)) & \text{ae } t \geq 0 \\ \hat{x}(0) = z_0 \in C, \end{cases} \quad (53)$$

satisfies, for some $\rho > 0$,

$$\lim_{t \rightarrow \infty} \|\hat{x}(t; z_0) - x(t; x_0)\| = 0, \text{ for all } z_0 \in B(x_0, \rho) \cap C.$$

To solve this problem we consider the Lipschitz mapping $\tilde{f} : H \times H \rightarrow H \times H$, defined as

$$\tilde{f}(z, x) := \left(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(x)), f(x) \right), \quad (54)$$

together with the set $S := C \times C$; hence, $N_S^P(x, y) = N_C(x) \times N_C(y)$, for every $(x, y) \in S$, so that S is also an r -uniformly prox-regular set. Consequently, we easily check that $y(t) := (\hat{x}(t; z_0), x(t; x_0))$ is the unique solution of the differential inclusion

$$\dot{y}(t) \in \tilde{f}(y(t)) - N_S(y(t)), \text{ ae } t \geq 0, y(0) = (z_0, x_0) \in S.$$

We have the following result, which extends [23, Proposition 3.5] in the case where the set C does not depend on the time variable.

Theorem 6.2 *Fix $(z_0, x_0) \in C \times C$ and assume that the solution of (1), $x(t; x_0)$, is bounded, say $\|x(t; x_0)\| \leq m$ for all $t \geq 0$. If $M := \sup\{\|f(x)\|, x \in B(\theta, m)\}$, we choose a Lipschitz continuous mapping \mathcal{L} together with positive numbers $\delta, \varepsilon, \eta > 0$ such that $\varepsilon < \delta r - M$, $\eta \leq (6\kappa)^{-1}\varepsilon$, and*

$$\|x - y\| \leq 3\eta, \ x, y \in H \implies \|\mathcal{L}(\mathcal{G}(x)) - \mathcal{L}(\mathcal{G}(y))\| \leq \varepsilon, \quad (55)$$

at the same time as, for all $x, y \in B(\theta, m + 3\eta)$,

$$\langle x - y, (f - \mathcal{L} \circ \mathcal{G})(x) - (f - \mathcal{L} \circ \mathcal{G})(y) \rangle \leq -\delta \|x - y\|^2. \quad (56)$$

Then for every $z_0 \in B(x_0, \eta)$ we have that

$$\|\hat{x}(t; z_0) - x(t; x_0)\| \leq e^{\frac{-(\delta - \frac{M + \varepsilon}{r})}{2} t} \|z_0 - x_0\|,$$

and, consequently,

$$\|\hat{x}(t; z_0) - x(t; x_0)\| \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Proof. For every $z, y \in B(\theta, m + 3\eta) \cap C$ such that $\|z - y\| \leq 3\eta$ we have that

$$\max\{\|f(z)\|, \|f(y)\|\} \leq M + 3\eta\kappa \leq M + \frac{\varepsilon}{2},$$

$$\|\mathcal{L}(\mathcal{G}(z)) - \mathcal{L}(\mathcal{G}(y))\| \leq \varepsilon.$$

We consider the (C^1-) function $V : H \times H \rightarrow \mathbb{R}$ defined as $V(z, y) := \frac{1}{2}\|z - y\|^2$. If $\beta := \delta - \frac{M+\varepsilon}{r}$, then by definition (54), we obtain

$$\begin{aligned} & \langle V'(z, y), (\tilde{f}(z, y) - N_S(z, y))^0 \rangle + \beta V(z, y) \\ &= \langle z - y, f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)) - \Pi_{N_C(z)}(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))) \rangle \\ & \quad + \langle y - z, f(y) - \Pi_{N_C(y)}(f(y)) \rangle + \frac{\beta}{2}\|z - y\|^2 \\ &= \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)) \rangle + \langle z - y, \Pi_{N_C(y)}(f(y)) \rangle \\ & \quad - \langle z - y, \Pi_{N_C(z)}(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))) \rangle + \frac{\beta}{2}\|z - y\|^2. \end{aligned}$$

Since $\Pi_{N_C(y)}(f(y)) \in N_C(y)$ and $\|\Pi_{N_C(y)}(f(y))\| \leq \|f(y)\|$, and similarly for $\Pi_{N_C(z)}(f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)))$, the last equality yields

$$\begin{aligned} & \langle V'(z, y), (\tilde{f}(z, y) - N_S(z, y))^0 \rangle + \beta V(z, y) \\ & \leq \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)) \rangle + \frac{\|f(y)\|}{2r}\|z - y\|^2 \\ & \quad + \frac{\|f(z) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y))\|}{2r}\|z - y\|^2 + \frac{\beta}{2}\|z - y\|^2, \end{aligned}$$

which by assumptions (55) and (56) gives us

$$\begin{aligned} & \langle V'(z, y), (\tilde{f}(z, y) - N_S(z, y))^0 \rangle + \beta V(z, y) \\ & \leq \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)) \rangle + \frac{\|f(z)\| + \|f(y)\|}{2r}\|z - y\|^2 \\ & \quad + \frac{\|\mathcal{L}(\mathcal{G}(z)) - \mathcal{L}(\mathcal{G}(y))\|}{2r}\|z - y\|^2 + \frac{\beta}{2}\|z - y\|^2 \\ & \leq \langle z - y, f(z) - f(y) - \mathcal{L}(\mathcal{G}(z)) + \mathcal{L}(\mathcal{G}(y)) \rangle + \frac{M + \varepsilon}{r}\|z - y\|^2 + \frac{\beta}{2}\|z - y\|^2 \\ & \leq -\delta\|z - y\|^2 + \left(\frac{M + \varepsilon}{r} + \frac{\beta}{2}\right)\|z - y\|^2 \leq 0. \end{aligned} \tag{57}$$

Now we choose $z_0 \in B(x_0, \eta) \cap C$, so that

$$B(z_0, \eta) \times B(x_0, \eta) \subset [B(\theta, m+3\eta) \times B(\theta, m+3\eta)] \cap \{(z, y) \in H \times H : \|z - y\| \leq 3\eta\}.$$

Then, thanks to (57), we can apply Corollary 5.1(i) to find some $t_0 > 0$ such that for every $t \in [0, t_0]$

$$e^{\beta t} V(\hat{x}(t; z_0), x(t; x_0)) \leq V(z_0, x_0);$$

that is,

$$\|\hat{x}(t; z_0) - x(t; x_0)\| \leq e^{\frac{-\beta t}{2}} \|z_0 - x_0\|.$$

Moreover, since $\|\hat{x}(t_0; z_0) - x(t_0; x_0)\| \leq \eta$ and $\hat{x}(t_0; z_0) \in B(\theta, m + 2\eta) \cap C$, we can also find $t_1 > 0$ such that for any $t \in [0, t_1]$

$$\begin{aligned} \|\hat{x}(t + t_0; z_0) - x(t + t_0; x_0)\| & \leq e^{\frac{-\beta t}{2}} \|\hat{x}(t_0; z_0) - x(t_0; x_0)\| \\ & \leq e^{\frac{-\beta t}{2}} e^{\frac{-\beta t_0}{2}} \|z_0 - x_0\| = e^{\frac{-\beta(t+t_0)}{2}} \|z_0 - x_0\|. \end{aligned}$$

Consequently, we deduce that for every $t \geq 0$

$$\|\hat{x}(t; z_0) - x(t; x_0)\| \leq e^{\frac{-\beta t}{2}} \|z_0 - x_0\|,$$

which completes the proof. ■

To close this section we consider the special case of linear Luenberger-like observers, where the assumption of Theorem 6.2 takes a simpler form. In this case (53) is written as

$$\begin{cases} \dot{\hat{x}}(t) \in (A - LG)\hat{x}(t) + LGx(t) - N_C(\hat{x}(t)) & \text{a.e. } t \geq 0 \\ \hat{x}(0) = z_0 \in C, \end{cases}$$

where $A, L, G : H \rightarrow H$ are linear continuous mappings; A^* and G^* will denote the corresponding adjoints mappings. Assume that $x(\cdot) := x(\cdot; x_0)$, $x_0 \in C$, is the solution of (1) (corresponding to $f = A$).

Corollary 6.2 *Fix $(z_0, x_0) \in C \times C$ and assume that the solution of (1) (corresponding to $f = A$), $x(t; x_0)$, is bounded, say $\|x(t; x_0)\| \leq m$ for all $t \geq 0$. Let $\delta, \varepsilon, \rho > 0$ be such that*

$$r^{-1}(m\|f\| + \varepsilon) < \delta, \text{ and } \frac{1}{2}(A + A^*) - \rho G^* G \leq -\delta \text{Id}.$$

If $L := \rho G^$, $\eta := \min\{(6\|A\|)^{-1}\varepsilon, (3\|LG\|)^{-1}\varepsilon\}$, and $\beta := \delta - r^{-1}(m\|A\| + \varepsilon)$, then for every $z_0 \in B(x_0, \eta)$ we have that, for all $t \geq 0$,*

$$\|\hat{x}(t; z_0) - x(t; x_0)\| \leq e^{\frac{-\beta t}{2}} \|z_0 - x_0\|.$$

Proof. The proof is similar as the one of Theorem 6.2, by observing that for every $x \in H$, we have

$$\langle x, (A - LG)x \rangle = \frac{\langle x, (A - LG)x \rangle + \langle x, (A^* - G^* L^*)x \rangle}{2}.$$

■

7 Concluding remarks

In this paper, we proved that a differential variational inequality involving a prox-regular set can be equivalently written as a differential inclusion governed by a maximal monotone operator. Therefore, the existence result and the stability analysis can be conducted in a classical way. We also give a characterization of lower semi-continuous α -Lyapunov pairs and functions. An application to a Luenberger-like observers is proposed. These new results will open new perspectives from both the numerical and applications points of view. An other interesting problem dealing with sweeping processes was introduced by J.J. Moreau in the seventies, which is of a great interest in applications. This problem is

obtained by replacing the fixed set C by a moving set $C(t)$, $t \in [0, T]$. It will be interesting to extend the ideas developed in this current work to the sweeping process involving prox-regular sets. Many other issues require further investigation including the study of numerical methods for problem (1) and the extension to second-order dynamical systems. This is out of the scope of the present paper and will be the subject of a future project of research.

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